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A Number-Theoretic Problem about Energy Levels of a Perturbed Harmonic Oscillator

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The determination of the degeneracy of a given energy level of an N-dimensional isotropic quantum oscillator that is perturbed by an isotropic quartic potential energy term leads to the question to enumerate the number of nonnegative integer solutions $(n_1,..., n_N)$, $n_1 \geqslant \cdots \geqslant n_N$, of the system $n = n_1 + \cdots + n_N$, $m = n_1^2 + \cdots + n_N^2$, as was shown recently by Louck and Metropolis. The present paper shows the partial reduction of this question to a similar one which is solved in the literature, precisely for $N \leqslant 8$, asymptotically for $N > 8$.

As shown in [4], the study of the energy spectrum of an N-dimensional isotropic quantum oscillator that is perturbed by an isotropic quartic potential energy term leads to the problem of enumerating the elements of the degeneracy-set ($n, m \in \mathbb{N}$ (always including 0))

$$E_N(n, m) := \{(n_1,..., n_N) \in \mathbb{N}^N \mid n_1^2 + \cdots + n_N^2 = m, \\ n_1 + \cdots + n_N = n, n_1 \geqslant n_2 \geqslant \cdots \geqslant n_N \geqslant 0\}.$$

The problem is solved for $N = 3$ in [4, 7] and an algorithmic procedure for general N is developed in [5]. A more complete partial solution will be given here using results on the number $r_N(n, m)$ of solutions of the Diophantine equations

$$x_1^2 + \cdots + x_N^2 = m, x_1 + \cdots + x_N = n \qquad (x_i \in \mathbb{Z}),$$

which is known precisely for $N \leqslant 8$ (see [1]) and asymptotically for general N (see [10, 3]).

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Let us describe the connection between $e_N(n, m) := \# E_N(n, m)$ and $r_N(n, m)$. Define

$$f_N(n, m) := \# \{(n_1, \ldots, n_N) \in \mathbb{N}^N \mid n_1^2 + \cdots + n_N^2 = m, n_1 + \cdots + n_N = n\}.$$

Evidently $e_N(n, m) \leqslant f_N(n, m)$ and

$$f_N(n, m) \leqslant N!\, e_N(n, m)$$

with equality when all n_i $(i = 1, \ldots, N)$ are distinct. To see the connection between $f_N(n, m)$ and $r_N(n, m)$, let us describe the situation geometrically: $r_N(n, m)$ is the number of "lattice points" $(x_1, \ldots, x_N) \in \mathbb{Z}^N$ in $\mathbb{R}^N$ lying on the intersection of the N-dimensional hypersphere $x_1^2 + \cdots + x_N^2 = m$ with the hyperplane $x_1 + \cdots + x_N = n$. (Thus $r_N(n, m) = 0$ for $n^2 > Nm$.) $f_N(n, m)$ enumerates only the lattice points with all coordinates nonnegative: $(x_1, \ldots, x_N) \in \mathbb{N}^N$. Easily we get

$$f_N(n, m) = \begin{cases} 0 & \text{for} \quad n^2 < m \\ r_N(n, m) & \text{for} \quad n^2 \geqslant (N-1)m. \end{cases}$$

For $m < n^2 < (N-1)m$ the situation is not so clear. So we have the following formulas

$$\frac{1}{N!} r_N(n, m) \leqslant e_N(n, m) \leqslant r_N(n, m) \qquad \text{for} \quad n^2 \geqslant (N-1)m,$$

$$e_N(n, m) = 0 \qquad \text{for} \quad n^2 < m \text{ and for } n^2 > Nm.$$

For $r_N(n, m)$ exact formulas are known in the literature for $N \leqslant 8$ (see [1]), for example, the following: Define $\Delta := Nm - n^2$ and suppose $\Delta \geqslant 0$. Suppose the necessary condition $n \equiv m \bmod 2$ is fulfilled.

$N = 3$.

$$r_3(n, m) = \begin{Bmatrix} 6 \\ 3 \end{Bmatrix} \cdot \sum_{d(\Delta/2)} \left(\frac{d}{3}\right) \qquad \text{for} \quad \begin{cases} n \equiv 0 \bmod 3, \\ n \not\equiv 0 \bmod 3. \end{cases}$$

$N = 4$.

$$r_4(n, m) = \begin{Bmatrix} 1 \\ \frac{1}{2} \end{Bmatrix} \cdot r_3(\Delta) \qquad \text{for} \quad \begin{cases} n \equiv m \equiv 0 \bmod 2, \\ n \equiv m \equiv 1 \bmod 2, \end{cases}$$

where $r_3(\Delta)$ is the number of representations of Δ as a sum of three squares. If $\Delta > 3$ is square free, then

$$r_4(n, m) = 12 h(\mathbb{Q}(\sqrt{-\Delta})),$$

where $h(\mathbb{Q}(\sqrt{-\Delta}))$ denotes the ideal class number of the imaginary quadratic numberfield $\mathbb{Q}(\sqrt{-\Delta})$ (see, e.g., [2, p. 175]). Consequently because $\frac{1}{24}r_4(n, m) \leqslant e_4(n, m) \leqslant r_4(n, m)$ for $n^2 \geqslant 3m$, the case $e_4(n, m) = 1$ (which is of special interest, in this case being no "higher degeneracy;" see [4]) can in the case of square free Δ and with $n^2 \geqslant 3m$ at most occur when $\mathbb{Q}(\sqrt{-\Delta})$ has class number 1 or 2, which is the case for 27 well-known imaginary quadratic fields, namely for $\mathbb{Q}(\sqrt{-\Delta})$ with

$$\Delta = 1, 2, 3, 5, 6, 7, 10, 11, 13, 15, 19, 22, 35, 37, 43, 51, 58,$$
$$67, 91, 115, 123, 163, 187, 235, 267, 403, 427.$$

The case Δ non square free can be handled similarly.

$N = 5$.

$$r_5(n, m) = 5C\frac{\Delta}{2} \sum_{d|(\Delta/2)} \left(\frac{d}{5}\right)\frac{1}{d},$$

where

$$C = \begin{cases} 1 & \text{for } \dfrac{\Delta}{2} \not\equiv 0 \bmod 5, \\ 1 - \left(\dfrac{\Delta_1}{5}\right) 5^{-a} & \text{for } 5^a \,\Big\|\, \dfrac{\Delta}{2}, a > 0, \dfrac{\Delta}{2} = 5^a \Delta_1. \end{cases}$$

$N = 6, 7, 8$ see [1],

For $N > 8$ no exact formulas seem to be known (see [8]), but asymptotic formulas were given by Valfisz [10]:

$$r_N(n, m) = \frac{\pi^{(N-1)/2}}{N^{(1/2)N-1}\Gamma\left(\dfrac{N-1}{2}\right)} \Delta^{(N-3)/2} S_N(n, m) + O(m^{(1/2)N-2} \log m),$$

where $S_N(n, m)$ is the so-called singular series, which can be computed explicitly; see [3]. Using these explicit formulas one sees that for N fixed

$$e_N(n, m) \to \infty \qquad \text{for} \quad \Delta = Nm - n^2 \to \infty((N-1)m < n^2 < Nm).$$

There is a generalization of the Diophantine system under consideration, namely

$$q(x_1, ..., x_N) = m, \qquad l(x_1, ..., x_N) = n,$$

where q is a given positive definite integral quadratic form and l a given integral linear form, which has been studied recently [9, 6]. It would be interesting to consider the question, whether there are relevant problems in physics, where this Diophantine system appears.

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